

LATTICES AND NORTON ALGEBRAS OF JOHNSON, GRASSMANN AND HAMMING GRAPHS

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ABSTRACT. To each of the Johnson, Grassmann and Hamming graphs we associate a lattice and characterize the eigenspaces of the adjacency operator in terms of this lattice. We also show that each level of the lattice induces in a natural way a tight frame for each eigenspace. For the most important eigenspace we compute explicitly the constant associated to the tight frame. Using the lattice we also give a formula for the product of the Norton algebra attached to that eigenspace.

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1. INTRODUCTION

Distance regular graphs are important in Algebraic Combinatorics [1] and have been generalized into other combinatorial objects such as association schemes [10, 20]. Some classical examples include the Johnson, Grassmann and Hamming graphs. Diverse algebras are associated to them, see for instance the Terwilliger Algebra in [4, 5, 6, 12, 16, 18, 25]. Another algebra involved to such schemes is the Norton Algebra. In the 1970's Norton constructed some commutative nonassociative algebras (called “Norton Algebras” by Conway and by Smith in [21]), whose automorphism groups contain finite groups generated by 3-transpositions, and in [7] this notion of algebra was applied to the case of an algebra constructed on the eigenspaces of the adjacency operator of an association scheme. (As is well known, related to this, Griess constructed the Monster simple group [11] as the automorphism group of a commutative nonassociative algebra of dimension $196883+1$. This algebra is known as the Monster algebra but also as the Conway-Griess-Norton algebra.)

In a recent work [19], we have studied the Norton algebra (in the sense of [7]) related to the dual polar graphs. While studying this problem we realized that the construction of a lattice associated to these spaces was helpful and that it has some interesting properties of its own. In particular the eigenspaces of the adjacency operator of the graph can be reconstructed from the lattice (see below). We wanted to extend these lattice results to the case of the Johnson, Grassmann and Hamming graphs, since there are some technical differences between them and the dual polar graphs. Using this framework, we also study their Norton Algebras.

Let X be the set of vertices of these graphs. The adjacency operator \mathbf{A} of the set of functions $\mathbb{R}^X = \{f : X \rightarrow \mathbb{R}\}$ induced by the distance on the graph gives a decomposition of \mathbb{R}^X into eigenspaces of \mathbf{A} .

We construct a graded lattice associated to the graph and characterize the eigenspaces of \mathbf{A} in terms of this lattice (Theorem 4.16).

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We show that the elements of each level of the lattice induces in a natural way a tight frame for each eigenspace (Theorem 5.4). For references about the theory of finite normalized tight frames see for example [3, 9, 14, 15, 26, 27].

The eigenspace V_1 corresponding to the second largest eigenvalue of \mathbf{A} is of particular importance since one can reconstruct the whole graph from the projections of the canonical basis onto it. We explicitly compute the constant of the tight frame attached to V_1 . (Proposition 5.5)

We use these and other constants associated to the lattice to give a formula for the product of the Norton algebra attached to V_1 . (Theorem 6.6).

This article is organized as follows: In Section 2 we give definitions. In Section 3, we define the lattice. In section 4, we give a convenient description for the eigenspaces V_i of \mathbf{A} . The technical Proposition 4.13 is crucial for the proof of Theorem 4.16. In Section 5 we obtain tight frames and calculate the different constants associated to them for each of the cases Johnson, Grassmann and Hypercube. In Section 6 we compute the Norton product using these constants.

2. DEFINITIONS

2.1. Distance regular graphs and their Adjacency algebras. [2]

Given $\Gamma = (X, E)$ a graph with distance $d(\cdot, \cdot)$ we say that it is distance regular if for any $(x, y) \in X \times X$ such that $d(x, y) = h$ and for all $i, j \geq 0$ the cardinal of the set $\{z \in X \mid d(x, z) = i \text{ and } d(y, z) = j\}$ is a constant denoted by p_{ij}^h which is independent of the pair (x, y) .

Let $\Gamma = (X, E)$ be a distance regular graph of diameter d . Let $Mat_X(\mathbb{R})$ denote the \mathbb{R} -algebra of matrices with real entries, where the rows and columns are indexed by the elements of X . For $0 \leq i \leq d$, the i th adjacency matrix of Γ is: $(A_i)_{xy} = \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{if } d(x, y) \neq i \end{cases}$. It is easy to see that the adjacency matrices satisfy:

(i') $A_0 = I$ where I is the identity matrix; (ii') $A_0 + \dots + A_d = J$ where J is the all 1's matrix; (iii') $A_i A_j = \sum_{h=0}^d p_{ij}^h A_h$ ($0 \leq i, j \leq d$); (iv') $A_i^t = A_i$. Thus A_0, \dots, A_d form a basis for a subalgebra \mathcal{A} of $Mat_X(\mathbb{R})$ called the *adjacency algebra* of Γ .

Recall that there exists a decomposition $\mathbb{R}^X = \oplus_{j=0}^d W_j$ where $\{W_j\}_{j=0}^d$ are common eigenspaces of $\{A_i\}_{i=0}^d$. Let $p_i(j)$ the eigenvalue of A_i on the eigenspace W_j . By Proposition 1.1 of section 3.1 of Chapter III of [2], $\{A_i\}_{i=0}^d$ and the eigenvalues $\{p_i(j)\}_{i,j=0}^d$ of a given Γ satisfy: $A_i = v_i(A_1)$, $p_i(j) = v_i(\theta_j)$, where $\theta_j = p_1(j)$, and $\{v_i\}_{i=0}^d$ are polynomials of degree i . We will order the decomposition according to $\theta_0 > \theta_1 > \dots > \theta_d$. In Theorem 5.1 of III.5 of [2], one can find formulas for the polynomials associated to each Γ .

We will use the standar notations concerning the space of functions \mathbb{R}^X :

i) $\mathbf{0}$ will denote the constant $\mathbf{0}(x) = 0, \forall x \in X$, ii) the same for the constant $\mathbf{1}$, iii) $\langle f, g \rangle := \sum_{x \in X} f(x)g(x)$, iv) $\|f\|^2 := \langle f, f \rangle$, v) for $U \subseteq \mathbb{R}^X$, $U^\perp = \{f \in \mathbb{R}^X : \langle f, g \rangle = 0 \forall g \in U\}$.

In addition, for ease of writing, we will use the following notation due to Iverson and Knuth ([17, 13]).

Notation 2.1. (*Iverson Bracket*)

For any statement P , let $[P] = \begin{cases} 1 & \text{if } P \text{ is true.} \\ 0 & \text{if } P \text{ is false.} \end{cases}$

Definition 2.2. Let $\mathbf{A} : \mathbb{R}^X \mapsto \mathbb{R}^X$ denote the adjacency operator defined by

$$\mathbf{A}(f)(x) = \sum_{y \in X} [d(x, y) = 1] f(y) = \sum_{y \in X: d(x, y) = 1} f(y)$$

Observe that $\mathbf{A}(f)(x) = \sum_{y \in X} (A_1)_{xy} f(y)$. Then $\{W_j\}_{j=0}^d$ are eigenspaces of \mathbf{A} with θ_j as corresponding eigenvalues.

\mathbf{A} is symmetric and it holds that $\langle \mathbf{A}(f), g \rangle = \langle f, \mathbf{A}(g) \rangle$.

2.2. Johnson, Grassman and Hypercube graphs.

We define the distances regular graphs that we will use in the rest of the paper.[1]

Johnson graph The vertex set of $J(n, k) = (X, E)$ ($2k \leq n$) is the set of all k -subsets of $[n] = \{1, \dots, n\}$, two vertices $x, y \in X$ being adjacent if and only if $|x \cap y| = k - 1$ and as a consequence $d(x, y) = j \Leftrightarrow |x \cap y| = k - j$. $J(n, k)$ has diameter $d = k$.

Grassman graph Let V be an n -dimensional vector space over a field F of q elements. The vertex set of $J_q(n, k) = (X, E)$ is the collection of linear subspaces of V of dimension k . Two vertices $x, y \in X$ are adjacent if and only if $\dim(x \cap y) = k - 1$ and clearly $d(x, y) = j \Leftrightarrow \dim(x \cap y) = k - j$. $J_q(n, k)$ has diameter $d = k$.

Hypercube graph Take S a set with two elements. The vertex set of $H(n, 2) = (X, E)$ is $S^n = \times_{i=1}^n S$ the cartesian product of n copies of S , two vertices $x, y \in X$ being adjacent if and only if they differ precisely in one coordinate and therefore $d(x, y) = j \Leftrightarrow x$ and y differ precisely in j coordinates. $H(n, 2)$ has diameter $d = n$.

2.3. Lattice.

We recall the following definitions (see [22]) and in the next section we associate a lattice to each one of the distance regular graphs previously defined.

- A partial order is a binary relation " \leq " over a set P which is reflexive, antisymmetric, and transitive.
- A partially ordered set (poset) (P, \leq) is a set P with a partial order \leq .
- A lattice $(\mathbf{L}, \leq, \wedge, \vee)$ is a poset (\mathbf{L}, \leq) in which every pair of elements $u, w \in \mathbf{L}$ has a least upper bound and a greatest lower bound. The first is called the join and it is denoted by $u \vee w$ and the second is called the meet and it is denoted by $u \wedge w$.
- A bounded lattice has a greatest (or maximum) and a least (or minimum) element, denoted $\hat{1}$ and $\hat{0}$ by convention.

3. LATTICE ASSOCIATED WITH JOHNSON, GRASSMAN AND HYPERCUBES GRAPHS

3.1. Johnson graph $J(n, k)$

- For $j = 0, \dots, k$ let Ω_j be the vertex set of $J(n, j)$ and $\Omega_{k+1} := \{\hat{1}\}$ where $\hat{1} = [n]$
- $\mathbf{L} = \cup_{\ell=0}^{k+1} \Omega_\ell$ and for $x, y \in \mathbf{L}$, $x \leq y \Leftrightarrow x \subseteq y$.
- With that order \mathbf{L} is a lattice with: $x \wedge y = x \cap y$, $x \vee y = \begin{cases} x \cup y & \text{if } |x \cup y| \leq k \\ \hat{1} & \text{if not} \end{cases}$

3.2. Grassman graph $J_q(n, k)$

- For $j = 0, \dots, k$ let Ω_j be the vertex set of $J_q(n, j)$ and $\Omega_{k+1} := \{\hat{1}\}$ where $\hat{1} = V$
- $\mathbf{L} = \cup_{\ell=0}^{k+1} \Omega_\ell$ and for $x, y \in \mathbf{L}$, $x \leq y \Leftrightarrow x \subseteq y$
- With that order \mathbf{L} is a lattice with:

$$x \wedge y = x \cap y, \quad x \vee y = \begin{cases} \text{span}\{x \cup y\} & \text{if } \dim(\text{span}\{x \cup y\}) \leq k \\ \hat{1} & \text{if not} \end{cases}$$

3.3. The Hypercube $H(n, 2)$ has as vertex set all words of length n with symbols taken from a set of 2 elements. We will take as our set of two elements the set $\{1, -1\}$, (instead of looking at words of 1's and 0's, as is traditional).

Let e_i be the vector with n coordinates that has a 1 in position i and 0 elsewhere, and let $f_i = -e_i$. Then, each word in $H(n, 2)$ is simply a sum of some e_i 's ($i \in I$) and some f_j 's, ($j \in J$) with the only restrictions on I and J that $I \cap J = \emptyset$ and $I \cup J = [n]$. Then the lattice associated is the following:

- For $0 \leq \ell \leq n$ we set $\Omega_\ell = \{\sum_{i \in I} e_i + \sum_{j \in J} f_j : I \cap J = \emptyset, |I \cup J| = \ell\}$. Given $x \in \Omega_\ell$, we represent $x = (I_x, J_x)$ where $x = \sum_{i \in I_x} e_i + \sum_{j \in J_x} f_j$. $I_x \cap J_x = \emptyset$ and $|I_x \cup J_x| = \ell$.

For $x, y \in \cup_{\ell=0}^n \Omega_\ell$, we define $x \leq y \Leftrightarrow I_x \subseteq J_x$ and $I_y \subseteq J_y$.

Observe that in the previous two cases Ω_ℓ is a member of the family of association schemes to which X belongs even when $\ell < n$. This does not happen in this case since the words in Ω_ℓ have 1, -1 and 0 in their entries while those in X have only 1's and -1's.

- Notice that $\Omega_0 := \{(0, 0, \dots, 0)\}$ and we add a dummy element $\hat{1}$ above all other elements, defining $\Omega_{n+1} := \{\hat{1}\}$, that is $x \leq \hat{1}, \forall x \in \cup_{\ell=0}^n \Omega_\ell$.

- With that order the set $\mathbf{L} = \cup_{\ell=0}^{n+1} \Omega_\ell$ is a lattice with:

$$x \wedge y = (I_x \cap I_y, J_x \cap J_y), \quad x \vee y = \begin{cases} (I_x \cup I_y, J_x \cup J_y) & \text{if } (I_x \cup I_y) \cap (J_x \cup J_y) = \emptyset \\ \hat{1} & \text{otherwise} \end{cases}$$

(for $x, y \in \mathbf{L} - \{\hat{1}\}$. Obviously $x \wedge \hat{1} = x, x \vee \hat{1} = \hat{1}$)

Definition 3.1.

(1) Recall ([22]) that given elements u, w of a poset one says that u covers w ; or w is covered by u , if $w < u$ but there is no z such that $w < z < u$. We denote it by $u > w$ or $w < u$.

(2) A bounded lattice is ranked if the poset \mathbf{L} is equipped with a rank function $\text{rk} : \mathbf{L} \rightarrow \mathbb{Z}$ compatible with the ordering (so $\text{rk}(u) \leq \text{rk}(w)$ whenever $u \leq w$) and such that if w covers u then $\text{rk}(w) = \text{rk}(u) + 1$. In our cases the lattices are clearly ranked, the rank of $w \in \Omega_\ell$ being ℓ .

(3) An atom is an element that covers $\hat{0}$ and a coatom is an element covered by $\hat{1}$. For $\Gamma = (X, E)$ any of the considered graphs, the set of atoms is Ω_1 and the set of coatoms is Ω_d where “ d ” is the diameter of Γ . Since in fact the set of coatoms Ω_d is the set of vertices X we will use both notations.

Is not difficult to prove that for each graph defined above, $(\mathbf{L}, \leq, \wedge, \vee)$ is a finite, bounded, ranked lattice with lowest element $\hat{0}$ and greatest element $\hat{1}$.

Remark 3.2. In all the cases: $d(x, y) = j \Leftrightarrow x \wedge y \in \Omega_{d-j}$.

Proof:

$$d(x, y) = j \Leftrightarrow \left\{ \begin{array}{ll} |x \cap y| = k - j & \text{for } J(n, k) \\ \dim(x \cap y) = k - j & \text{for } J_q(n, k) \\ x \text{ and } y \text{ have } n - j & \\ \text{coordinates in common} & \text{for } H(n, 2) \end{array} \right\} \Leftrightarrow x \wedge y \in \Omega_{d-j}.$$

QED.

Lemma 3.3. The lattice \mathbf{L} has the following properties:

- (1) \mathbf{L} is atomic, that is every element of the lattice is a join of atoms.
- (2) $\forall u, w \in \mathbf{L}$ such that $u \vee w \neq \hat{1} \Rightarrow \text{rk}(u) + \text{rk}(w) = \text{rk}(u \vee w) + \text{rk}(u \wedge w)$

Proof:

(1) In the Johnson case, each element of the lattice is a set of elements taken from $[n]$, so if $z = \{i_1, \dots, i_j\}$, then $z = \{i_1\} \vee \{i_2\} \vee \dots \vee \{i_j\}$ is a join of atoms.

In the Grassman case, each element z of the lattice is a subspace of $GF(q)^n$, so taking a basis $\{v_1, \dots, v_j\}$ of z , we obtain that $z = \text{span}(v_1) \vee \text{span}(v_2) \vee \dots \vee \text{span}(v_j)$ is a join of atoms.

In the Hamming case, $\hat{1} = e_1 \vee f_1$ and an element $z \neq \hat{1}$ of the lattice is of the form $z = \sum_{i \in I_z} e_i + \sum_{j \in J_z} f_j$ with $I_z \cap J_z = \emptyset$, so $z = \bigvee_{i \in I_z} e_i \vee \bigvee_{j \in J_z} f_j$ is a join of atoms.

(2) In the Johnson case, $\text{rk}(z)$ is the cardinality of z , so the previous formula is simply the inclusion-exclusion formula for 2 sets: $|A| + |B| = |A \cup B| + |A \cap B|$.

In the Grassman, the rank of an element is the dimension, so the formula is true because of the well known identity $\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W)$.

In the Hamming case, $\text{rk}(z) = |I_z| + |J_z|$ so again the formula is true because of the inclusion-exclusion formula for sets. QED.

Corollary 3.4.

If τ and σ are different atoms such that $\tau \vee \sigma \neq \hat{1}$, then $\text{rk}(\tau \vee \sigma) = 2$.

Lemma 3.5. Let u and w be elements of the lattice which are not coatoms, then $(u \vee w > u, w) \Rightarrow (u, w > u \wedge w)$.

Reciprocally $(u, w > u \wedge w \text{ and } u \vee w \neq \hat{1}) \Rightarrow u \vee w > u, w$.

Proof: In order to prove the first statement, observe that $z > w \Leftrightarrow z \geq w$ and $\text{rk}(z) = \text{rk}(w) + 1$. So, $u \vee w > u$ and $u \vee w > w \Leftrightarrow \text{rk}(u \vee w) = \text{rk}(u) + 1 = \text{rk}(w) + 1$ (in particular, we must have that $\text{rk}(u) = \text{rk}(w)$). Also, since u and w are not coatoms and $\text{rk}(u \vee w) = \text{rk}(u) + 1$ we deduce that $u \vee w \neq \hat{1}$. Then, by Lemma 3.3 (2), we get $\text{rk}(u) + \text{rk}(w) - \text{rk}(u \wedge w) = \text{rk}(u) + 1$, i.e., $\text{rk}(w) = \text{rk}(u \wedge w) + 1$, which implies that $w > u \wedge w$. The proof is similar for u .

Reciprocally, if $u > u \wedge w$ and $w > u \wedge w \Rightarrow \text{rk}(u) = \text{rk}(w) = \text{rk}(u \wedge w) + 1$. Using Lemma 3.3 (2) we get $\text{rk}(w) = \text{rk}(u) + \text{rk}(w) - \text{rk}(u \vee w) + 1$ which implies $\text{rk}(u) + 1 = \text{rk}(u \vee w)$ and then that $u \vee w$ covers u (and similarly w). QED.

4. DESCRIPTION OF THE EIGENSPACES USING THE ASSOCIATED LATTICE

In this section let $\Gamma = (X, E)$ be any of the distance regular graphs of diameter d already defined ($J(n, k)$, $J_q(n, k)$ or $H(n, 2)$), together with its associated decomposition: $\mathbb{R}^X = \bigoplus_{i=0}^d W_i$, where $\{W_i\}_{i=0}^d$ are the common eigenspaces of the adjacency matrices of Γ .

We will describe each of the eigenspaces $\{W_i\}_{i=0}^d$, using the lattice previously defined. The description give us a recursive formulae for the eigenvalues $\{\theta_j\}_{j=0}^d$ associated to each graph $\Gamma = (X, E)$ defined in Subsection 2.2.

Definition 4.1. For $z \in \Omega_j$, define:

$$a_j^\ell = \begin{cases} \{y \in \Omega_\ell : z \leq y\} & \text{if } j \leq \ell \\ \{y \in \Omega_\ell : y \leq z\} & \text{if } j > \ell \end{cases} \quad a_j = |\{x \in X : z \leq x\}|$$

Note that $a_j = a_j^d$ if $j \leq d$, but $a_{d+1} = 0$.

The previous definitions seem to depend on z , we show next this is not so.

Recall that $\binom{i}{j}_q$ is the number of j -dimensional subspaces of $GF(q)^i$. A formula

is given by: $\binom{i}{j}_q = \frac{[i]_q [i-1]_q \dots [i-j+1]_q}{[j]_q [j-1]_q \dots [1]_q}$ where $[i]_q = \begin{cases} \frac{q^i - 1}{q - 1} & \forall i \geq 1 \\ 0 & \forall i < 1 \end{cases}$,

Lemma 4.2.

$$a_j^\ell = \begin{cases} \binom{j}{\ell} & \text{for } J(n, k) \\ \binom{j}{\ell}_q & \text{for } J_q(n, k) \text{ if } \ell \leq j \\ \binom{j}{\ell} & \text{for } H(n, 2) \end{cases} \quad a_j^\ell = \begin{cases} \binom{n-j}{\ell-j} & \text{for } J(n, k) \\ \binom{n-j}{\ell-j}_q & \text{for } J_q(n, k) \text{ if } \ell \geq j \\ 2^{\ell-j} \binom{n-j}{\ell-j} & \text{for } H(n, 2) \end{cases}$$

Proof: Given $z \in \Omega_j$ and $\ell \leq j$, we have to count the elements of the set $\{y \in \Omega_\ell : y \leq z\}$. Looking at the construction of the lattice, the lemma follows straightforward in this case.

In the case $j \leq \ell$ the proof is also easy for $J(n, k)$ and $J_q(n, k)$. For the case $H(n, 2)$, if we fix $z \in \Omega_j$ and we count the elements of $\{y \in \Omega_\ell : z \leq y\}$, we have to choose $\ell - j$ coordinates from the $n - j$ not used by z , and we can fill each of them with 1 or -1 . QED.

Definition 4.3.

$\iota : \mathbf{L} \rightarrow \mathbb{R}^X : z \mapsto \iota_z$ is the map defined by $\iota_z(x) = [z \leq x] \forall z \in \mathbf{L}, x \in X$.

Lemma 4.4.

For $j = 0, 1, \dots, d$ and $\forall z, y \in \mathbf{L}$,

$$i) \iota_z = \mathbf{0} \Leftrightarrow z = \hat{1} \quad ii) \iota_z = \mathbf{1} \Leftrightarrow z = \hat{0} \quad iii) \|\iota_z\|^2 = a_j \quad \forall z \in \Omega_j$$

$$iv) \iota_z \iota_y = \iota_{z \vee y} \quad v) \langle \iota_z, \iota_y \rangle = \|\iota_{z \vee y}\|^2$$

Proof:

$i), ii)$ and $iii)$ are easy to prove. For $iv)$

$$\iota_z(x) \iota_y(x) = [z \leq x][y \leq x] = [z \vee y \leq x] = \iota_{z \vee y}(x)$$

To prove $v)$, observe that

$$\langle \iota_z, \iota_y \rangle = \sum_{x \in X} \iota_z(x) \iota_y(x) = \sum_{x \in X} \iota_{z \vee y}(x) = \sum_{x \in X} (\iota_{z \vee y}(x))^2 = \|\iota_{z \vee y}\|^2$$

QED.

Corollary 4.5.

- (i) $z \vee y = \hat{1}$ if and only if ι_z and ι_y are orthogonal to each other.
- (ii) If $z \vee y \in \Omega_j$, then $\langle \iota_z, \iota_y \rangle = a_j$

$$(iii) \text{ If } \tau \text{ and } \sigma \text{ are both atoms then } \langle \iota_\tau, \iota_\sigma \rangle = \begin{cases} a_1 & \text{if } \tau = \sigma \\ 0 & \text{if } \tau \vee \sigma = \hat{1} \\ a_2 & \text{otherwise} \end{cases}$$

4.1. A filtration for \mathbb{R}^X .**Definition 4.6.**

For $j = 0, 1, \dots, d$, let $\Lambda_j \subseteq \mathbb{R}^X$ be the subspace generated by $\{\iota_x\}_{x \in \Omega_j}$.

We want to show that $\Lambda_j \subseteq \Lambda_{j+1}$. That is, they form a filtration for \mathbb{R}^X . We need some tools first.

Definition 4.7. Given $w \in \mathbf{L}$, let:

$$w^* = \sum_v [v > w] \iota_v \quad w_* = \sum_v [v < w] \iota_v$$

Lemma 4.8. Given $w \in \Omega_j \subseteq \mathbf{L}$

- (1) $w^* = c_j \iota_w$ where c_j only depends on $j = \text{rk}(w)$
- (2) $w_* = a_j^{j-1} \iota_w + \Phi_w$ where $\Phi_w : X \rightarrow \{0, 1\}$, $\Phi_w(x) = [w \wedge x \in \Omega_{j-1}]$

Proof: (1) Given $x \in X$, We have:

$$\begin{aligned} w^*(x) &= \sum_v [v \cdot > w][v \leq x] = |\{v \in \Omega_{j+1} : w \leq v \leq x\}| \\ &= \begin{cases} 0 & \text{if } w \not\leq x \\ \begin{cases} k-j & \text{for } J(n, k) \\ [k-j]_q & \text{for } J_q(n, k) \\ n-j & \text{for } H(n, 2) \end{cases} & \text{if } w \leq x \end{cases} \end{aligned}$$

The last equality is easy to prove for Johnson and Grassman graphs. In the Hamming case, if $w = \sum_{i \in I_w} e_i + \sum_{j \in J_w} f_j$ and $x = \sum_{i \in I_x} e_i + \sum_{j \in J_x} f_j$ to build v we need to add to the sum constituting w one e_i with $i \in I_x - I_w$ or else one f_j with $j \in J_x - J_w$. So:

$$\begin{aligned} |\{v \in \Omega_{j+1} : w \leq v \leq x\}| &= |I_x - I_w| + |J_x - J_w| \\ &= |I_x| + |J_x| - (|I_w| + |J_w|) = n - j \end{aligned}$$

That is $w^* = c_j \iota_w$ and the constant c_j only depends on the $\text{rk}(w) = j$.

To prove the identity (2), given $x \in X$ we have

$$\begin{aligned} w_*(x) &= \sum_v [w \cdot > v] \iota_v(x) = |v : v \leq w \text{ and } v \leq x| \\ &\stackrel{(\dagger)}{=} \begin{cases} a_j^{j-1} & \text{if } w \leq x \text{ (equivalently } \text{rk}(w \wedge x) = j) \\ 1 & \text{if } \text{rk}(w \wedge x) = j-1 \\ 0 & \text{if } \text{rk}(w \wedge x) < j-1 \end{cases} \end{aligned}$$

Thus $w_* = a_j^{j-1} \iota_w + \Phi_w$. The proof of (\dagger) follows from the fact in the case $\text{rk}(w \wedge x) < j-1$ there cannot be any such v . This is because if $v \leq x$ and $v \leq w$ then $v = v \wedge x \leq w \wedge x$ and $j-1 = \text{rk}(v) \leq \text{rk}(w \wedge x)$. QED.

Corollary 4.9. $\Lambda_0 \subseteq \Lambda_1 \subseteq \dots \subseteq \Lambda_d = \mathbb{R}^X$

Proof: It follows from definition of Λ_j and part (1) of previous lemma. QED.

Definition 4.10. Let $V_0 = \Lambda_0$ and $V_j = \Lambda_j \cap \Lambda_{j-1}^\perp$ $j = 1, \dots, d$.

We have that $\Lambda_j = V_0 \oplus V_1 \oplus \dots \oplus V_j = \Lambda_{j-1} \oplus V_j$. We want to show that for $j = 1, \dots, d$, $V_j \neq \{0\}$, that is $\Lambda_{j-1} \neq \Lambda_j$. To prove this, we need more lemmas. Recall (Definition 2.2) that the operator \mathbf{A} is $\mathbf{A}(f)(x) = \sum_{y \in X} [d(x, y) = 1]f(y)$.

Lemma 4.11. If $x \in X$, then $\mathbf{A}(\iota_x) = \sum_{y \in X} [d(x, y) = 1] \iota_y$.

Proof: Note that for $x, y \in X$ we have $\iota_x(y) = [x = y]$ Thus, if $z \in X$:

$$\begin{aligned} \mathbf{A}(\iota_x)(z) &= \sum_{y \in X} [d(z, y) = 1] \iota_x(y) = \sum_{y \in X} [d(z, y) = 1][x = y] = [d(z, x) = 1] \\ &= \sum_{y \in X} [d(x, y) = 1][y = z] = \left(\sum_{y \in X} [d(x, y) = 1] \iota_y \right)(z) \end{aligned}$$

QED.

Lemma 4.12. Let $x \in X$. Then $x_* = \mathbf{A}(\iota_x) + \iota_x a_d^{(d-1)}$.

Proof: Note that for $x, y \in X = \Omega_d$ we have by the previous proof that $\mathbf{A}(\iota_x)(y) = [d(x, y) = 1] = [x \wedge y \in \Omega_{d-1}] = \Phi_x(y)$, where Φ is as in Lemma 4.8. Hence, the result follows from that lemma. QED.

Proposition 4.13. For each $j < d$ there are constants α_j, β_j such that if $w \in \Omega_j$ then:

$$\sum_{\tilde{w} \succ w} u_* = \alpha_j \iota_w + \beta_j w_*$$

The constants are:

$$\alpha_j = \begin{cases} n-2j & J(n, k) \\ [n-2j]_q q^j & J_q(n, k) \\ 2(n-j) & H(n, 2) \end{cases}, \quad \beta_j = \begin{cases} c_{j-1} & J(n, k), J_q(n, k) \\ c_{j-1} - 1 & H(n, 2) \end{cases}$$

where c_j was given in the proof of Lemma 4.8.

Proof:

$$\begin{aligned} \sum_{u \triangleright w} u_* &= \sum_{u \triangleright w} \sum_{z \triangleleft u} \iota_z = \sum_{u \in \Omega_{j+1}} \sum_{z \in \Omega_j} [z \vee w \leq u] \iota_z \\ &= \sum_{u \triangleright w} \iota_w + \sum_{u \in \Omega_{j+1}} \sum_{z \in \Omega_j - w} [z \vee w = u] \iota_z \\ &= a_j^{j+1} \iota_w + \sum_{z \in \Omega_j} [\text{rk}(z \vee w) = j+1] \iota_z \quad (1) \end{aligned}$$

$$\text{Similar arguments show that } \sum_{v \triangleleft w} v^* = a_j^{j-1} \iota_w + \sum_{z \in \Omega_j} [\text{rk}(z \wedge w) = j-1] \iota_z \quad (2)$$

For $J(n, k)$ and $J_q(n, k)$ it follows from Lemma 3.3 (2) that $\text{rk}(z \vee w) = j+1 \Leftrightarrow \text{rk}(z \wedge w) = j-1$ so the sums at the rightmost side in (1) and (2) are equal and

$$\sum_{u \triangleright w} u_* - \sum_{v \triangleleft w} v^* = (a_j^{j+1} - a_j^{j-1}) \iota_w$$

By Lemma 4.8 $\sum_{v \triangleleft w} v^* = \sum_{v \triangleleft w} c_{j-1} \iota_v = c_{j-1} w_*$, thus in the Johnson and Grassman cases we have $\sum_{u \triangleright w} u_* = (a_j^{j+1} - a_j^{j-1}) \iota_w + c_{j-1} w_*$. The values of $\alpha_j = a_j^{j+1} - a_j^{j-1}$ and $\beta_j = c_{j-1}$ follow from Lemmas 4.2 and 4.8.

The case of $H(n, 2)$ is different because it is easy to prove that in this case

$$\begin{aligned} &\{z \in \Omega_j : \text{rk}(z \wedge w) = j-1\} = \\ &\{z \in \Omega_j : \text{rk}(z \vee w) = j+1\} \cup \{z \in \Omega_j : \text{rk}(z \wedge w) = j-1 \text{ and } z \vee w = \hat{1}\}, \\ \therefore \sum_{u \triangleright w} u_* - \sum_{v \triangleleft w} v^* &= (a_j^{j+1} - a_j^{j-1}) \iota_w \\ &\quad - \underbrace{\sum_{z \in \Omega_j} [\text{rk}(z \wedge w) = j-1] [z \vee w = \hat{1}] \iota_z}_{\Psi_w}. \quad (3) \end{aligned}$$

Now given $x \in X$ we will evaluate $\Psi_w(x)$.

$$\begin{aligned} \Psi_w(x) &= \sum_{z \in \Omega_j} [\text{rk}(z \wedge w) = j-1] [z \vee w = \hat{1}] [z \leq x] \\ &= |\{z \in \Omega_j : \text{rk}(z \wedge w) = j-1, z \vee w = \hat{1} \text{ and } z \leq x\}|. \end{aligned}$$

Suppose that there is such a z . Such z must be unique: since $z \wedge w \in \Omega_{j-1}$ then it must be $w = (z \wedge w) \vee \sigma$, where $\sigma \in \Omega_1$. Similarly $z = (z \wedge w) \vee \tau$ for some τ . Since $z \vee w = \hat{1}$ it must be $\tau = -\sigma$. Thus z is uniquely defined if it exists.

Moreover, since $(z \leq x \Rightarrow w \wedge z \leq w \wedge x)$ then we must have $j-1 \leq \text{rk}(w \wedge x)$. But $\text{rk}(w \wedge x) = j \Leftrightarrow w \leq x$ and since $z \leq x$ this implies that $z \vee w \leq x$ which is an absurd since $z \vee w = \hat{1}$. So $\text{rk}(w \wedge x) = j-1$ and $w \wedge x = w \wedge z$. Thus $w \wedge x \in \Omega_{j-1}$ must hold if z exists. Conversely, if $w \wedge x \in \Omega_{j-1}$ and $w = (x \wedge w) \vee \sigma$ then $z = (x \wedge w) \vee (-\sigma)$ satisfies all the conditions. Thus, we conclude that $\Psi_w(x) = [w \wedge x \in \Omega_{j-1}] = \Phi_w(x)$. Hence by Lemma 4.8 (2) equation (3) becomes

$$\begin{aligned} \sum_{u \triangleright w} u_* - \sum_{v \triangleleft w} v^* &= (a_j^{j+1} - a_j^{j-1}) \iota_w - \Phi_w \\ &= (a_j^{j+1} - a_j^{j-1}) \iota_w - (w_* - a_j^{j-1} \iota_w) = a_j^{j+1} \iota_w - w_*. \end{aligned}$$

As before, $\sum_{v \triangleleft w} v^* = c_{j-1} w^*$, thus $\sum_{u \triangleright w} u^* = a_j^{j+1} \iota_w + (c_{j-1} - 1) w^*$. Again the values of $\alpha_j = a_j^{j+1}$ and $\beta_j = c_{j-1} - 1$ follow from Lemmas 4.2 and 4.8. QED.

Lemma 4.14.

For $0 \leq j \leq d$, there exist a constant λ_j such that $\mathbf{A}(v) - \lambda_j v \in \Lambda_{j-1}$, $\forall v \in \Lambda_j$.

Proof: It is enough to prove it for elements of the spanning set $\{\iota_u\}_{u \in \Omega_j}$. The proof is by reverse induction on the levels of the lattice, starting at $j = d$. The inductive hypothesis will be:

There exists constants λ_j and ν_j such that $\mathbf{A}(\iota_u) = \lambda_j \iota_u + \nu_j u^*$ for all $u \in \Omega_j$.

This will prove the lemma since by Definition 4.7, $u^* \in \Lambda_{j-1}$.

The inductive hypothesis is true for $j = d$ since if $x \in \Omega_d$ then by Lemma 4.12 $\mathbf{A}(\iota_x) = -a_d^{(d-1)} \iota_x + x^*$. Now assume the hypothesis true for $j + 1$ and let us prove it for j . Let $w \in \Omega_j$. By Lemma 4.8 $\iota_w = \frac{1}{c_j} w^* = \frac{1}{c_j} \sum_{u \triangleright w} \iota_u$, thus

$$\begin{aligned} \mathbf{A}(\iota_w) &= \frac{1}{c_j} \sum_{u \triangleright w} \mathbf{A}(\iota_u) = \frac{1}{c_j} \sum_{u \triangleright w} (\lambda_{j+1} \iota_u + \nu_{j+1} u^*) \\ &\stackrel{(4.13)}{=} \frac{\lambda_{j+1}}{c_j} w^* + \frac{\nu_{j+1}}{c_j} (\alpha_j \iota_w + \beta_j w^*) \\ &\stackrel{(*)}{=} \left(\lambda_{j+1} + \frac{\nu_{j+1} \alpha_j}{c_j} \right) \iota_w + \frac{\nu_{j+1} \beta_j}{c_j} w^* \end{aligned}$$

QED.

Corollary 4.15. For $j = 0, \dots, d$, Λ_j are \mathbf{A} -invariant subspaces of \mathbb{R}^X .

Proof: This follows directly by the previous Lemma and Corollary 4.9. QED.

Theorem 4.16. For $j = 0, \dots, d$, $V_j = \Lambda_j \cap \Lambda_{j-1}^\perp$ are the eigenspaces W_j of \mathbf{A} given in 2.1, in that order. The corresponding eigenvalues θ_j are the λ_j 's given in Lemma 4.14.

Proof: Take $v \in V_j (\subseteq \Lambda_j)$. By Lemma 4.14 $\mathbf{A}(v) = \lambda_j v + v'$ with $v' \in \Lambda_{j-1}$ and by Corollary 4.15 $\mathbf{A}(v') \in \Lambda_{j-1}$. Then by definition of V_j :

$$\begin{aligned} 0 &= \langle v, \mathbf{A}(v') \rangle = \langle \mathbf{A}(v), v' \rangle = \langle \lambda_j v + v', v' \rangle \\ &= \langle \lambda_j v, v' \rangle + \langle v', v' \rangle = \|v'\|^2 \end{aligned}$$

thus $\mathbf{A}(v) = \lambda_j v \forall v \in V_j$. Therefore, $\mathbb{R}^X = \bigoplus_{j=0}^d V_j$ where each V_j is either zero or an eigenspace of \mathbf{A} . Since $X = \Omega_d$ is the set of vertices of $\Gamma = (X, E)$; a distance regular graph of diameter d ; there are exactly $d + 1$ eigenspaces of the adjacency matrix A_1 , therefore of \mathbf{A} . Thus each V_j is indeed an eigenspace of \mathbf{A} (hence $V_j \neq 0 \forall j$) and λ_j are the eigenvalues of \mathbf{A} .

From the proof of Lemma 4.14 (identity $(*)$) we get that $\nu_d = 1$ and the recursion $\nu_j = \frac{\nu_{j+1} \beta_j}{c_j} \forall j < d$. From the values of the constants, it follows that $\nu_j = c_{j-1}$ in the Johnson and Grassman cases and $\nu_j = 1$ in the Hamming case. Therefore we conclude that $\lambda_j = \lambda_{j+1} + \alpha_j$ in the first two cases and $\lambda_j = \lambda_{j+1} + \frac{\alpha_j}{c_j} = \lambda_{j+1} + 2$ in the latter case. Therefore it is clear that $\lambda_0 > \lambda_1 > \dots > \lambda_d$. This imply (by the ordering of 2.1) that $\lambda_j = \theta_j$. QED.

Remark 4.17. From the recursion of the λ 's and the fact that $\lambda_j = \theta_j$, we obtain that the eigenvalues of \mathbf{A} satisfy the following recursive formulae:

$$\theta_d = \begin{cases} -k & \text{for } J(n, k) \\ -[k]_q & \text{for } J_q(n, k) \\ -n & \text{for } H(n, 2) \end{cases} \quad \theta_j = \begin{cases} \theta_{j+1} + n - 2j & \text{for } J(n, k) \\ \theta_{j+1} + [n - 2j]_q q^j & \text{for } J_q(n, k) \\ \theta_{j+1} + 2 & \text{for } H(n, 2) \end{cases}$$

hence they are:

$$\theta_j = \begin{cases} (k-j)(n-k-j) - j & \text{for } J(n, k) \\ q^{j+1}[k-j]_q[n-k-j]_q - [j]_q & \text{for } J_q(n, k) \\ n - 2j & \text{for } H(n, 2) \end{cases}$$

The formulae for θ_j can be founded in the literature (see Chapter 9 of [1]). The proof above gives another way to compute them.

5. TIGHT FRAMES FOR THE EIGENSPACES

In this section we will consider $\Gamma = (X, E)$ any of the graphs already defined, \mathbf{L} the associated lattice described in Section 3 and $\mathbb{R}^X = \bigoplus_{j=0}^d V_j$ the corresponding decomposition. We will prove that each Ω_j induces a finite tight frame on each V_j via the map defined in 4.3. We will give a formula for the constants associated to these tight frames and in the case of the eigenspace of the second largest eigenvalue we will compute explicitly the constant associated.

Definition 5.1.

For $j = 0, 1, \dots, d$; let π_j be the orthogonal projection $\pi_j : \mathbb{R}^X \rightarrow V_j$. Then for each $u \in \Omega_j$, denote $\tilde{u}^j = \pi_j(\iota_u)$. Since the set $\{\iota_u\}_{u \in \Omega_j}$ span Λ_j , the projections $\{\tilde{u}^j\}_{u \in \Omega_j}$ span V_j . When it is obvious from the context we will denote it by \tilde{u} .

Proposition 5.2. For $j = 0, 1, \dots, d$, let $U^j \in \mathbb{R}^{X \times X}$ be the matrix

$$(U^j)_{x,y} = (x, y)^j = \sum_{u \in \Omega_j} \iota_u(x) \iota_u(y).$$

Then for every $j = 0, \dots, d$, \tilde{u}^j is an eigenvector of U^j with eigenvalue $\mu_j = \sum_{i=0}^{d-j} a_{d-i}^j p_i(j)$ where $p_i(j)$ are the eigenvalues of A_i (the i -th adjacency matrix of the graph) corresponding to the eigenspace V_j .

Proof: Let $(x, y) \in X \times X$ and $l = \text{rk}(x \wedge y)$.

$$\begin{aligned} U_{x,y}^j &= (x, y)^j = \sum_{u \in \Omega_j} \iota_u(x) \iota_u(y) = \sum_{u \in \Omega_j} [u \leq x][u \leq y] = \sum_{u \in \Omega_j} [u \leq x \wedge y] \\ &= \begin{cases} |\{u \in \Omega_j : u \leq x \wedge y\}| & \text{if } j \leq l \\ 0 & \text{if } j > l \end{cases} = \begin{cases} a_l^j & \text{if } j \leq l \\ 0 & \text{if } j > l \end{cases} \end{aligned}$$

This and Remark 3.2 shows that $U^j = \sum_{l=j}^d a_l^j A_{d-l}$. Then, as $\tilde{u}^j \in V_j$ is an eigenvector of the adjacency matrices, we have that for every $0 \leq j \leq d$, \tilde{u}^j is an eigenvector of U^j with eigenvalue $\mu_j = \sum_{l=j}^d a_l^j p_{d-l}(j)$.

Making the change of variable $i = d - l$, we have: $\mu_j = \sum_{i=0}^{d-j} a_{d-i}^j p_i(j)$. QED.

Definition 5.3. Let V be a finite vector space with inner product \langle, \rangle . A finite tight frame on V is a finite set $F \subseteq V$ which satisfies the following condition: there exists a non-zero constant μ such that:

$$\sum_{v \in F} |\langle f, v \rangle|^2 = \mu \|f\|^2 \quad \forall f \in V.$$

Theorem 5.4. For $j = 0, \dots, d$ and for all $f \in V_j$, it holds

$$\sum_{u \in \Omega_j} \langle \tilde{u}^j, f \rangle \tilde{u}^j = \mu_j f$$

where μ_j is the eigenvalue of Proposition 5.2.

In particular the set $\{\tilde{u}^j\}_{u \in \Omega_j}$ is a finite tight frame for V_j .

Proof: Let $w, v \in \Omega_j$.

$$\begin{aligned}
\langle \mu_j \check{w}^j, \check{v}^j \rangle &= \sum_{x \in X} \mu_j \check{w}^j(x) \check{v}^j(x) \stackrel{5.2}{=} \sum_{x \in X} \left(\sum_{y \in X} (x, y)^j \check{w}^j(y) \right) \check{v}^j(x) \\
&= \sum_{x, y \in X} \sum_{u \in \Omega_j} \iota_u(x) \iota_u(y) \check{w}^j(y) \check{v}^j(x) \\
&= \sum_{u \in \Omega_j} \langle \iota_u, \check{w}^j \rangle \langle \iota_u, \check{v}^j \rangle = \sum_{u \in \Omega_j} \langle \check{u}^j, \check{w}^j \rangle \langle \check{u}^j, \check{v}^j \rangle \\
&= \langle \sum_{u \in \Omega_j} \langle \check{u}^j, \check{w}^j \rangle \check{u}^j, \check{v}^j \rangle.
\end{aligned}$$

Since this is true for an arbitrary elements of the spanning set $\{\check{v}^j\}_{v \in \Omega_j}$ of V_j then $\mu_j \check{w}^j = \sum_{u \in \Omega_j} \langle \check{u}^j, \check{w}^j \rangle \check{u}^j$, and again since this holds for arbitrary w then it holds for any element of V_j . QED.

5.1. Computation of μ_1 . In the following we give a more explicit calculation of μ_1 , the constant associated to the tight frame corresponding to V_1 ; the second largest eigenspaces of $\Gamma = (X, E)$

Proposition 5.5.

$$\mu_1 = \begin{cases} \binom{n-2}{k-1} & \text{for } J(n, k) \\ \binom{n-2}{k-1}_q q^{k-1} & \text{for } J_q(n, k) \\ 2^{n-1} & \text{for } H(n, 2) \end{cases}$$

Proof: By Proposition 5.2 we have that $\mu_1 = \sum_{i=0}^{d-1} a_{d-i}^1 p_i(1)$. One can find the following formulae for $p_i(1)$ in pages 220 of [2] for $J(n, k)$; 262, 263, 302 of [2] for $J_q(n, k)$ and 210 of [2] for $H(n, 2)$;

$$p_i(1) = \begin{cases} \sum_{t=0}^i (-1)^t \binom{1}{t} \binom{k-1}{i-t} \binom{n-k-1}{i-t} & \text{for } J(n, k) \\ \left(1 - \frac{[i]_q [n]_q}{[k]_q [n-k]_q q^i}\right) q^{i^2} \binom{k}{i}_q \binom{n-k}{i}_q & \text{for } J_q(n, k) \\ \binom{n}{i} - 2 \binom{n-1}{i-1} & \text{for } H(n, 2) \end{cases}$$

Then by Proposition 5.2:

• For $J(n, k)$:

$$\begin{aligned}
\mu_1 &= \sum_{i=0}^{k-1} a_{k-i}^1 p_i(1) = \sum_{i=0}^{k-1} (k-i) \left(\sum_{t=0}^i (-1)^t \binom{1}{t} \binom{k-1}{i-t} \binom{n-k-1}{i-t} \right) \\
&= \sum_{i=0}^{k-1} (k-i) \left(\binom{k-1}{i} \binom{n-k-1}{i} - \binom{k-1}{i-1} \binom{n-k-1}{i-1} \right)
\end{aligned}$$

Defining $b_i = \binom{k-1}{i} \binom{n-k-1}{i}$ (hence $b_i = 0$ for $i < 0$).

$$\begin{aligned}
&= \sum_{i=0}^{k-1} k(b_i - b_{i-1}) - \sum_{i=0}^{k-1} i(b_i - b_{i-1}) \\
&= k b_{k-1} - (b_1 - b_0 + 2(b_2 - b_1) + 3(b_3 - b_2) + \dots) \\
&= k b_{k-1} - \left(- \sum_{i=0}^{k-1} b_i + k b_{k-1} \right) = \sum_{i=0}^{k-1} b_i = \binom{n-2}{k-1}
\end{aligned}$$

(the last equation by the Chu-Vandermonde identity: $\sum_{i=0}^n \binom{s}{i} \binom{t}{n-i} = \binom{s+t}{n}$).

- For $J_q(n, k)$:

$$\begin{aligned}
\mu_1 &= \sum_{i=0}^{k-1} a_{k-i}^1 p_i(1) \\
&= \sum_{i=0}^{k-1} [k-i]_q \left(1 - \frac{[i]_q [n]_q}{[k]_q [n-k]_q q^i} \right) q^{i^2} \binom{k}{i}_q \binom{n-k}{i}_q \\
&= \sum_{i=0}^{k-1} [k-i]_q q^{i^2} \binom{k}{i}_q \binom{n-k}{i}_q - \sum_{i=0}^{k-1} \frac{[k-i]_q [i]_q [n]_q q^{i^2} \binom{k}{i}_q \binom{n-k}{i}_q}{[k]_q [n-k]_q q^i}
\end{aligned}$$

$$\text{Since: } \frac{[k-i]_q}{[k]_q} \binom{k}{i}_q = \binom{k-1}{i}_q = \binom{k-1}{k-1-i}_q,$$

$$\text{and: } \frac{[i]_q}{[n-k]_q} \binom{n-k}{i}_q = \binom{n-k-1}{i-1}_q = \binom{n-k-1}{n-k-i}_q,$$

$$\text{then: } \mu_1 = [k]_q \sum_{i=0}^{k-1} \binom{k-1}{k-1-i}_q \binom{n-k}{i}_q q^{i^2} - [n]_q \sum_{i=0}^{k-1} \binom{k-1}{i}_q \binom{n-k-1}{n-k-i}_q q^{i^2-i}$$

Using q -Vandermonde: $\sum_i \binom{m}{k-i}_q \binom{n}{i}_q q^{i(m-k+i)} = \binom{m+n}{k}_q$ and $[k]_q [n-1]_q = [n]_q [k-1]_q + [n-k]_q q^{k-1}$ we have:

$$\begin{aligned}
\mu_1 &= [k]_q \binom{n-1}{k-1}_q - [n]_q \binom{n-2}{n-k}_q = [k]_q \binom{n-1}{k-1}_q - [n]_q \binom{n-2}{k-2}_q \\
&= \binom{n-2}{k-2}_q \left([k]_q \frac{[n-1]_q}{[k-1]_q} - [n]_q \right) \\
&= \binom{n-2}{k-2}_q \frac{[n-k]_q}{[k-1]_q} q^{k-1} = \binom{n-2}{k-1}_q q^{k-1}
\end{aligned}$$

- For $H(n, 2)$,

$$\begin{aligned}
\mu_1 &= \sum_{i=0}^{n-1} a_{n-i}^1 p_i(1) = \sum_{i=0}^{n-1} (n-i) \left(\binom{n}{i} - 2 \binom{n-1}{i-1} \right) \\
&= \sum_{i=0}^{n-1} (n-i) \binom{n}{i} - 2 \sum_{i=0}^{n-1} (n-i) \binom{n-1}{i-1} \\
&= \sum_{i=0}^{n-1} n \binom{n-1}{i} - 2 \sum_{i=1}^{n-1} (n-1) \binom{n-2}{i-1} \\
&= n2^{n-1} - 2(n-1)2^{n-2} = 2^{n-1}
\end{aligned}$$

QED.

6. APPLICATION: NORTON PRODUCT ON V_1

Given the decomposition $\mathcal{R}^X = V_0 \oplus V_1 \oplus \dots \oplus V_d$, in this section we describe the product of a Norton algebra attached to the eigenspace V_1 .

Definition 6.1. *The Norton algebra on V_1 is the algebra given by the product $f \star g = \pi_1(fg)$ for $f, g \in V_1$.*

It is easy to check that it is a commutative, nonassociative algebra. We want to compute the \star product in V_1 for the graphs concerning on this paper. Since $\Lambda_1 = \text{span}\{\iota_\tau : \tau \in \Omega_1\}$ the set $\{\check{\tau}\}_{\tau \in \Omega_1}$ spans V_1 and we have proved in Theorem 5.4 that they are a finite tight frame for V_1 . We will describe $\check{\tau} \star \check{\sigma}$ in a simplified form using such a frame.

For this we need the following results.

Lemma 6.2. *For all $\tau \in \Omega_1$, $\check{\tau} = \iota_\tau - \frac{a_1}{|X|}\mathbf{1}$ with a_1 given in Lemma 4.2.*

Proof: Recall that $\langle \iota_{\check{0}} \rangle = \Lambda_0 \subseteq \Lambda_1 = \langle \{\iota_\tau\}_{\tau \in \Omega_1} \rangle$, and $\Lambda_1 = \Lambda_0 \oplus V_1$. Since $\forall \tau \in \Omega_1$, $\check{\tau} = \pi_1(\iota_\tau) \in V_1$, we have $\check{\tau} = \iota_\tau - t\mathbf{1}$ for some $t \in \mathbb{R}$.

From the fact that $\langle \check{\tau}, \mathbf{1} \rangle = 0$ we conclude $t = \frac{\langle \iota_\tau, \mathbf{1} \rangle}{\|\mathbf{1}\|^2} = \frac{\sum_{x \in X} [\tau \leq x]}{|X|} = \frac{a_1}{|X|}$. QED.

Proposition 6.3. *Let $h \in \mathbb{R}^X$, then $\pi_1(h) = \sum_{\tau \in \Omega_1} \frac{\langle \iota_\tau, h \rangle}{\mu_1} \check{\tau}$.*

Proof:

$$\begin{aligned} \mu_1 \pi_1(h) &\stackrel{5.4}{=} \sum_{\tau \in \Omega_1} \langle \check{\tau}, \pi_1(h) \rangle \check{\tau} \stackrel{\dagger}{=} \sum_{\tau \in \Omega_1} \langle \check{\tau}, h \rangle \check{\tau} \stackrel{6.2}{=} \sum_{\tau \in \Omega_1} \langle \iota_\tau - \frac{a_1}{|X|}\mathbf{1}, h \rangle \check{\tau} \\ &= \sum_{\tau \in \Omega_1} \langle \iota_\tau, h \rangle \check{\tau} - \langle \frac{a_1}{|X|}\mathbf{1}, h \rangle \sum_{\tau \in \Omega_1} \check{\tau} \stackrel{*}{=} \sum_{\tau \in \Omega_1} \langle \iota_\tau, h \rangle \check{\tau} \end{aligned}$$

(\dagger holds since $h = \pi_0(h) + \pi_1(h) + \dots + \pi_d(h)$, $\pi_i(h) \in V_i$ and $\langle V_i, V_j \rangle = 0 \forall i \neq j$;
 $*$ holds since $\sum_{\tau \in \Omega_1} \iota_\tau \in \Lambda_0 \Rightarrow \sum_{\tau \in \Omega_1} \check{\tau} = \mathbf{0}$) QED.

Lemma 6.4.

$$\check{\tau} \star \check{\sigma} = \pi_1(\iota_{\tau \vee \sigma}) - \frac{a_1}{|X|}(\check{\tau} + \check{\sigma})$$

Proof: Recall by Lemma 6.2 $\check{\tau} = \iota_\tau - \frac{a_1}{|X|}\mathbf{1}$. Then:

$$\begin{aligned} \check{\tau} \star \check{\sigma} &= \pi_1(\check{\tau} \check{\sigma}) = \pi_1\left(\left(\iota_\tau - \frac{a_1}{|X|}\mathbf{1}\right)\left(\iota_\sigma - \frac{a_1}{|X|}\mathbf{1}\right)\right) \\ &= \pi_1\left(\iota_\tau \iota_\sigma - \frac{a_1}{|X|}(\iota_\tau + \iota_\sigma) + \left(\frac{a_1}{|X|}\right)^2 \mathbf{1}\right) \\ &= \pi_1(\iota_\tau \iota_\sigma) - \frac{a_1}{|X|}\pi_1(\iota_\tau + \iota_\sigma) + \left(\frac{a_1}{|X|}\right)^2 \pi_1(\mathbf{1}) \\ &= \pi_1(\iota_{\tau \vee \sigma}) - \frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) \end{aligned}$$

QED.

Lemma 6.5. *If a_j are as in Lemma 4.2, then $\langle \iota_\rho, \iota_{\tau \vee \sigma} \rangle = a_{\text{rk}(\rho \vee \tau \vee \sigma)}$.*

Proof:

$$\begin{aligned} \langle \iota_\rho, \iota_{\tau \vee \sigma} \rangle &= \sum_{x \in X} \iota_\rho(x) \iota_{\tau \vee \sigma}(x) = \sum_{x \in X} [\rho \leq x][\tau \vee \sigma \leq x] \\ &= \sum_{x \in X} [\rho \vee \tau \vee \sigma \leq x] = |\{x \in X : \rho \vee \tau \vee \sigma \leq x\}| = a_{\text{rk}(\rho \vee \tau \vee \sigma)} \end{aligned}$$

QED.

Theorem 6.6. *For $H(n, 2)$, $\check{\tau} \star \check{\sigma} = \mathbf{0}$.*

$\check{\tau} \star \check{\tau} = (1 - \frac{2k}{n})\check{\tau}$ in the Johnson case and $(1 - \frac{2[k]_q}{[n]_q})\check{\tau}$ in the Grassmann case, while for $\tau \neq \sigma$:

$$\check{\tau} \star \check{\sigma} = \begin{cases} \frac{2k-n}{n(n-2)}(\check{\tau} + \check{\sigma}) & \text{For } J(n, k) \\ -\frac{[k]_q}{[n]_q}(\check{\tau} + \check{\sigma}) + \frac{[k-1]_q}{q[n-2]_q} \sum_{\rho \leq \tau \vee \sigma} \check{\rho} & \text{For } J_q(n, k) \end{cases}$$

Proof: By Lemma 6.4, if $\tau = \sigma$ we have that $\check{\tau} \star \check{\tau} = \check{\tau} - 2 \frac{a_1}{|X|} \check{\tau}$. Replacing a_1 by Lemma 4.2 the formulae follow straightforward for all the graphs.

For the case $\tau \neq \sigma$ we will use the notation $\Psi_j = \{\rho \in \Omega_1 : \text{rk}(\rho \vee \tau \vee \sigma) = j\}$. Using Lemmas 6.4 and 6.5:

$$\begin{aligned} \check{\tau} \star \check{\sigma} &= -\frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) + \pi_1(\iota_{\tau \vee \sigma}) \\ &= -\frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) + \sum_{\rho \in \Omega_1} \frac{\langle \iota_\rho, \iota_{\tau \vee \sigma} \rangle}{\mu_1} \check{\rho} = -\frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) + \sum_{\rho \in \Omega_1} \frac{a_{\text{rk}(\rho \vee \tau \vee \sigma)}}{\mu_1} \check{\rho} \\ &= -\frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) + \frac{a_2}{\mu_1} \sum_{\rho \in \Psi_2} \check{\rho} + \frac{a_3}{\mu_1} \sum_{\rho \in \Psi_3} \check{\rho} + \mathbf{0} \end{aligned}$$

The last zero since $a_{d+1} = 0$. Also, since:

$$\sum_{\rho \in \Psi_3} \check{\rho} = \sum_{\rho \in \Omega_1} \check{\rho} - \sum_{\rho \in \Psi_2} \check{\rho} - \sum_{\rho \in \Psi_{d+1}} \check{\rho}$$

and $\sum_{\rho \in \Omega_1} \iota_\rho \in \Lambda_0 \Rightarrow \sum_{\rho \in \Omega_1} \check{\rho} = \mathbf{0}$ we have then:

$$(\diamond) \quad \check{\tau} \star \check{\sigma} = -\frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) + \frac{a_2 - a_3}{\mu_1} \sum_{\rho \in \Psi_2} \check{\rho} - \frac{a_3}{\mu_1} \sum_{\rho \in \Psi_{d+1}} \check{\rho}$$

Then, we have, in each case:

- For $J(n, k)$, $\Psi_2 = \{\tau, \sigma\}$ and $\Psi_{d+1} = \emptyset$. In this case then (\diamond) becomes:

$$\begin{aligned} \check{\tau} \star \check{\sigma} &= -\frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) + \frac{a_2 - a_3}{\mu_1}(\check{\tau} + \check{\sigma}) \\ &= \left(-\frac{\binom{n-1}{k-1}}{\binom{n}{k}} + \frac{\binom{n-2}{k-2} - \binom{n-3}{k-3}}{\binom{n-2}{k-1}} \right) (\check{\tau} + \check{\sigma}) \\ &= \left(-\frac{k}{n} + \frac{k-1}{n-2} \right) (\check{\tau} + \check{\sigma}) = \frac{2k-n}{n(n-2)} (\check{\tau} + \check{\sigma}) \end{aligned}$$

- For $J_q(n, k)$, $\Psi_2 = \{\rho \in \Omega_1 : \rho \leq \tau \vee \sigma\}$ and $\Psi_{d+1} = \emptyset$.

Then (\diamond) becomes: $\check{\tau} \star \check{\sigma} = -\frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) + \frac{a_2 - a_3}{\mu_1} \sum_{\rho \leq \tau \vee \sigma} \check{\rho}$

Recall that in this case, $a_j = \binom{n-j}{k-j}_q$ (Lemma 4.2), $|X| = \binom{n}{k}_q$ and $\mu_1 = \binom{n-2}{k-1}_q q^{k-1}$ (Proposition 5.5).

$$\begin{aligned} \text{Thus } \check{\tau} \star \check{\sigma} &= -\frac{[k]_q}{[n]_q}(\check{\tau} + \check{\sigma}) + \frac{1 - \frac{[k-2]_q}{[n-2]_q}}{\frac{[n-k]_q}{[k-1]_q} q^{k-1}} \sum_{\rho \leq \tau \vee \sigma} \check{\rho} \\ &= -\frac{[k]_q}{[n]_q}(\check{\tau} + \check{\sigma}) + \frac{([n-2]_q - [k-2]_q)[k-1]_q}{[n-2]_q [n-k]_q q^{k-1}} \sum_{\rho \leq \tau \vee \sigma} \check{\rho} \\ &= -\frac{[k]_q}{[n]_q}(\check{\tau} + \check{\sigma}) + \frac{((q^{n-2} - 1) - (q^{k-2} - 1))(q^{k-1} - 1)}{(q^{n-2} - 1)(q^{n-k} - 1)q^{k-1}} \sum_{\rho \leq \tau \vee \sigma} \check{\rho} \\ &= -\frac{[k]_q}{[n]_q}(\check{\tau} + \check{\sigma}) + \frac{q^{k-2}(q^{n-k} - 1)(q^{k-1} - 1)}{(q^{n-2} - 1)(q^{n-k} - 1)q^{k-1}} \sum_{\rho \leq \tau \vee \sigma} \check{\rho} \\ &= -\frac{[k]_q}{[n]_q}(\check{\tau} + \check{\sigma}) + \frac{[k-1]_q}{[n-2]_q q} \sum_{\rho \leq \tau \vee \sigma} \check{\rho} \end{aligned}$$

• For $H(n, 2)$ $\Psi_2 = \{\tau, \sigma\}$ and $\Psi_{d+1} = \{-\tau, -\sigma\}$ and it holds that $\forall \rho \in \Omega_1$ $\iota_\rho + \iota_{-\rho} = \mathbf{1}$ therefore: $\check{\rho} + (-\rho) = \mathbf{0} \forall \rho \in \Omega_1$, i.e. $(-\rho) = -\check{\rho}$. Then:

$$\begin{aligned} \check{\tau} \star \check{\sigma} &= -\frac{a_1}{|X|}(\check{\tau} + \check{\sigma}) + \frac{a_2 - a_3}{\mu_1}(\check{\tau} + \check{\sigma}) - \frac{a_3}{\mu_1}((-\check{\tau}) + (-\check{\sigma})) \\ &= -\frac{2^{n-1}}{2^n}(\check{\tau} + \check{\sigma}) + \frac{2^{n-2} - 2^{n-3}}{2^{n-1}}(\check{\tau} + \check{\sigma}) - \frac{2^{n-3}}{2^{n-1}}(-\check{\tau} - \check{\sigma}) \\ &= -\frac{1}{2}(\check{\tau} + \check{\sigma}) + \frac{1}{2}(\check{\tau} + \check{\sigma}) = \mathbf{0} \end{aligned}$$

QED.

Remark 6.7. *The fact that in the Hamming case the Norton product reduces to zero can also be deduced from Theorem 5.2 of [7] since it can be shown that the “Krein parameters” $q_{1,1}^1$ are 0 in this case.*

7. CONCLUSION

For each of the Johnson, Grassmann and Hamming graphs we constructed a ranked (finite) lattice which we embed into \mathbb{R}^X (Definition 4.3). For the levels Ω_j the corresponding embeddings Λ_j in \mathbb{R}^X are shown to be a filtration, and we characterized the eigenspaces W_j of the adjacency operator in terms of these Λ_j s. (Theorem 4.16). We also show that each Ω_j induces in a natural way a tight frame for each eigenspace. Using the lattice we give a formula for the product of the Norton algebra attached to W_1 .

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